

# ON THE LOCAL SMOOTHING FOR THE SCHRÖDINGER EQUATION

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ABSTRACT. We prove a family of identities that involve the solution  $u$  to the following Cauchy problem:

$$i\partial_t u + \Delta u = 0, u(0) = f(x), (t, x) \in \mathbf{R}_t \times \mathbf{R}_x^n,$$

and the  $\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)$ -norm of the initial datum  $f$ . As a consequence of these identities we shall deduce a lower bound for the local smoothing estimate proved in [3], [8] and [9] and a uniqueness criterion for the solutions to the Schrödinger equation.

This paper is devoted to the study of the following Cauchy problem:

$$(0.1) \quad i\partial_t u + \Delta u = 0, u(0) = f(x), (t, x) \in \mathbf{R}_t \times \mathbf{R}_x^n, n \geq 1.$$

It is well-known that the solution to (0.1) satisfies the local smoothing estimate (see [3], [8] and [9]):

$$(0.2) \quad \sup_{R \in (0, \infty)} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\nabla u|^2 dx dt \leq C \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2 \quad \forall f \in \dot{H}^{\frac{1}{2}}(\mathbf{R}^n),$$

where  $B_R$  denotes the ball in  $\mathbf{R}^n$  centered in the origin of radius  $R$ ,  $\nabla$  denotes the gradient with respect to the space variables and  $\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)$  is the usual homogeneous Sobolev space.

Let us recall that the estimate (0.2) has played a crucial role in the study of the nonlinear Schrödinger equation with nonlinearities involving derivatives (see [4]).

Some questions can be raised in connection with the local smoothing stated above. It is natural to ask whether the l.h.s. in (0.2) can be bounded from below in the following way:

$$(0.3) \quad \sup_{R \in (0, \infty)} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\nabla u|^2 dx dt \geq c \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2,$$

where as usual  $u(t, x)$  solves (0.1) with initial datum  $f$  and  $c > 0$  is a suitable constant. Notice that an estimate of this type implies that (0.2) is an equivalence more than an inequality.

Another natural question connected with (0.2) concerns the behaviour at infinity of the following function:

$$F_f : (0, \infty) \ni R \rightarrow \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\nabla u|^2 dx dt \in (0, \infty),$$

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where as usual  $u(t, x)$  is the unique solution to (0.1). In fact, in the best of our knowledge, it is not known whether the following implication is true:

$$(0.4) \quad \liminf_{R \rightarrow \infty} F_f(R) = 0 \implies f = 0.$$

Notice that a positive answer to (0.4) gives a uniqueness criterion for the solutions to the Schrödinger equation.

As a by product of the results of this paper, we can deduce that (0.3) and (0.4) are true.

In order to state our basic result we have to fix some notations.

**Notation.** For any  $s \in \mathbf{R}$  and for any  $n \in \mathbf{N}$ , the spaces  $\dot{H}^s(\mathbf{R}^n)$  shall denote the homogeneous Sobolev spaces of order  $s$ , whose norm is defined as follows:

$$\|f\|_{\dot{H}^s}^2 = \int_{\mathbf{R}^n} |\hat{f}(\xi)|^2 |\xi|^{2s} d\xi,$$

where

$$\hat{f}(\xi) := \int_{\mathbf{R}^n} e^{-2\pi i x \xi} f(x) dx.$$

In the case  $s = 0$  we shall also use the notation  $\dot{H}^0(\mathbf{R}^n) = L^2(\mathbf{R}^n)$ .

We shall denote by  $\mathcal{S}(\mathbf{R}^n)$  the Schwartz functional space.

If  $f \in C^\infty(\mathbf{R}^n)$ , then  $\partial_r f$  and  $\nabla_\tau f$  denote respectively the radial derivative of  $f$  and the tangential part of the full gradient  $\nabla f$ .

If  $z = x + iy \in \mathbf{C}$  is a complex number, then  $\mathcal{R}e z$  and  $\mathcal{I}m z$  shall denote its real and imaginary part. We shall denote by  $\bar{z}$  its complex conjugate number, i.e.  $\bar{z} = x - iy$ .

For any  $R > 0$  we shall denote by  $B_R$  the ball of  $\mathbf{R}^n$  centered in the origin of radius  $R$ .

We can now state the basic theorem of this paper.

**Theorem 0.1.** *Let  $\psi \in C^\infty(\mathbf{R}^n)$  be a real-valued and radially symmetric function such that:*

(1) *the following estimates hold:*

$$|\partial_r \psi(|x|)|, |\partial_r^2 \psi(|x|)|, |\partial_r^3 \psi(|x|)| \leq C |P(|x|)| \quad \forall |x| \in \mathbf{R}^+$$

*where  $C > 0$  and  $P(|x|)$  is a polynomial;*

(2) *the following limit exists:*

$$\lim_{|x| \rightarrow \infty} \partial_r \psi(|x|) := \psi'(\infty) \in (-\infty, \infty).$$

*Then we have the following identity:*

$$(0.5) \quad \lim_{T \rightarrow \infty} \int_{-T}^T \int_{\mathbf{R}^n} \left[ \nabla \bar{u}(t, x) D^2 \psi(x) \nabla u(t, x) - \frac{1}{4} |u(t, x)|^2 \Delta^2 \psi(x) \right] dt dx \\ = 2\pi \psi'(\infty) \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2 \quad \forall f \in \mathcal{S}(\mathbf{R}^n),$$

*where  $u(t, x)$  is the unique solution to (0.1) with initial datum  $f$ ,  $D^2 \psi$  is the hessian matrix  $\left( \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right)_{i,j=1,\dots,n}$  and  $\Delta^2$  denotes the bilaplacian operator.*

*Remark 0.1.* Let us point out that already the existence of the limit in the l.h.s. in (0.5) is not a trivial fact, thus its existence must be considered as a part of the statement.

*Remark 0.2.* Let us recall that in [2] the authors were able to show an inequality between the l.h.s. and the r.h.s. of (0.5). Then, the main point in (0.5) is that it represents an identity and not only an inequality.

*Remark 0.3.* The identity (0.5) can be exploited in many ways. One possibility is to choose in (0.5) a function  $\psi$  such that: it satisfies all the assumptions of the theorem 0.1, it is convex and  $\Delta^2 \psi(x) \leq 0$  for any  $x \in \mathbf{R}^n$ . This was the main strategy used in [2] in order to prove the local smoothing estimate (0.2) in dimension  $n \geq 3$  and in presence of a potential type perturbation. Notice that in this way it is possible to give a proof of (0.2), at least in dimension  $n \geq 3$ , which does not involve the Fourier transform, that was a basic tool in [3], [8] and [9]. However in the sequel we shall exploit (0.5) in a different direction by taking the advantage of the fact that it is an identity.

*Remark 0.4.* Let us notice that the identities proved in [5] in dimension  $n \geq 3$ , follow from the general identity (0.5) by choosing  $\psi(|x|) = |x|$ . It is clear that this choice for the function  $\psi(|x|)$  is not a-priori allowed since  $x \rightarrow |x|$  is not a  $C^\infty$  function. However to overcome this difficulty we can choose in (0.5) the functions  $\psi(|x|) = \psi_\epsilon(|x|) = \sqrt{\epsilon^2 + |x|^2}$  and to take the limit for  $\epsilon \rightarrow 0$  in the corresponding identities.

As consequence of theorem 0.1 we get the following result in the spirit of those given in [1].

**Corollary 0.1.** *Assume that  $n \geq 1$  and  $u(t, x)$  solves (0.1) with initial datum  $f$ , then the following identity holds:*

$$(0.6) \quad \lim_{R \rightarrow \infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\partial_r u|^2 dx dt = 2\pi \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2 \quad \forall f \in \dot{H}^{\frac{1}{2}}(\mathbf{R}^n),$$

and in particular

$$(0.7) \quad \sup_{R \in (0, \infty)} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\nabla u|^2 dx dt \geq 2\pi \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2 \quad \forall f \in \dot{H}^{\frac{1}{2}}(\mathbf{R}^n).$$

If moreover we assume that  $u(t, x)$  satisfies the following condition:

$$(0.8) \quad \liminf_{R \rightarrow \infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\nabla u|^2 dx dt = 0,$$

then  $u(t, x) = 0$ .

*Remark 0.5.* Notice that the existence of the limit in (0.6) is not a trivial fact and it must be considered as a part of the statement.

The rest of the paper is organized as follows: section 1 is devoted to the proof of theorem 0.1 while in section 2 we shall prove corollary 0.1.

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## 1. PROOF OF THEOREM 0.1

Let us start this section with the following

**Lemma 1.1.** *If  $u(t, x)$  is the unique solution to (0.1) where  $f \in \mathcal{S}(\mathbf{R}^n)$ , and  $\psi(|x|)$  is as in theorem 0.1, then:*

$$(1.1) \quad \lim_{t \rightarrow \pm\infty} \mathcal{I}m \int_{\mathbf{R}^n} \bar{u}(t, x) \nabla \psi(x) \nabla u(t, x) dx = \pm 2\pi \psi'(\infty) \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2.$$

**Proof.** In the proof we shall need the following asymptotic formula for the solution  $u(t, x)$  to (0.1) with initial datum  $f \in \mathcal{S}(\mathbf{R}^n)$ :

$$(1.2) \quad \lim_{t \rightarrow \pm\infty} \left\| u(t, x) - e^{\mp i n \pi / 4} \frac{e^{\pm i \frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \hat{f} \left( \pm \frac{x}{4\pi t} \right) \right\|_{L^2(\mathbf{R}^n)} = 0,$$

see [7] for a proof.

We first compute the limit in (1.1) as  $t \rightarrow +\infty$ . By using (1.2) we get

$$\lim_{t \rightarrow \infty} \|u(t, x) - v(t, x)\|_{L^2(\mathbf{R}^n)} = 0,$$

where  $v(t, x) := e^{-i n \pi / 4} \frac{e^{i \frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} \hat{f} \left( \frac{x}{4\pi t} \right)$ .

On the other hand if  $u$  satisfies (0.1), then its partial derivatives  $\partial_j u$  are still solutions of (0.1) with Cauchy data  $\partial_j f$ , for any  $j = 1, \dots, n$ .

We can then apply again (1.2) in order to deduce the following fact:

$$\lim_{t \rightarrow \infty} \|\partial_j u(t, x) - w_j(t, x)\|_{L^2(\mathbf{R}^n)} = 0 \quad \forall j = 1, \dots, n,$$

where

$$w_j(t, x) := e^{-i n \pi / 4} \frac{e^{i \frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} i \frac{x_j}{2t} \hat{f} \left( \frac{x}{4\pi t} \right).$$

Notice that we have used the identity  $\widehat{\partial_j f}(\xi) = 2\pi i \xi_j \hat{f}(\xi)$ . We can now easily deduce that

$$\lim_{t \rightarrow \infty} \int_{\mathbf{R}^n} [\bar{u}(t, x) \partial_j u(t, x) - \bar{v}(t, x) w_j(t, x)] \phi_j(x) dx = 0$$

$$\forall \phi_j \in L^\infty(\mathbf{R}^n), j = 1, \dots, n.$$

In particular if we choose  $\phi_j(x) = \partial_r \psi(|x|) \frac{x_j}{|x|}$ , then we have:

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathcal{I}m \int_{\mathbf{R}^n} \bar{u}(t, x) \nabla \psi(x) \nabla u(t, x) dx \\ &= \lim_{t \rightarrow \infty} \mathcal{I}m \sum_{j=1}^n \int_{\mathbf{R}^n} \bar{u}(t, x) \partial_j u(t, x) \frac{x_j}{|x|} \partial_r \psi(|x|) dx \\ &= \lim_{t \rightarrow \infty} \mathcal{I}m \sum_{j=1}^n \int_{\mathbf{R}^n} \bar{v}(t, x) w_j(t, x) \frac{x_j}{|x|} \partial_r \psi(|x|) dx \\ &= \lim_{t \rightarrow \infty} \mathcal{I}m \frac{i}{(4\pi t)^n} \int_{\mathbf{R}^n} \frac{|x|}{2t} \left| \hat{f} \left( \frac{x}{4\pi t} \right) \right|^2 \partial_r \psi(|x|) dx. \end{aligned}$$

The previous chain of identities, combined with the change of variable formula, imply:

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathcal{I}m \int_{\mathbf{R}^n} \bar{u}(t, x) \nabla \psi(x) \nabla u(t, x) dx \\ &= 2\pi \lim_{t \rightarrow \infty} \int_{\mathbf{R}^n} |y| |\hat{f}(y)|^2 \partial_r \psi(4\pi t |y|) dy = 2\pi \psi'(\infty) \int_{\mathbf{R}^n} |y| |\hat{f}(y)|^2 dy. \end{aligned}$$

The limit as  $t \rightarrow -\infty$  in (1.1) can be computed in a similar way by exploiting (1.2) in the case  $t \rightarrow -\infty$ . □

**Proof of theorem 0.1.** Following [2] we multiply (0.1) by

$$\nabla \psi \nabla \bar{u} + \frac{1}{2} \Delta \psi \bar{u}$$

and we integrate by parts.

Let us start by writing the following identities:

$$\begin{aligned} & \mathbf{i} \partial_t u \left( \frac{1}{2} \Delta \psi \bar{u} + \nabla \psi \nabla \bar{u} \right) \\ &= \frac{\mathbf{i}}{2} [\operatorname{div}(\nabla \psi \bar{u} \partial_t u) - \bar{u} \nabla \psi \nabla \partial_t u - \partial_t u \nabla \psi \nabla \bar{u}] \\ & \quad + \mathbf{i} \partial_t (u \nabla \psi \nabla \bar{u}) - \mathbf{i} u \nabla \psi \nabla \partial_t \bar{u} \\ &= \frac{\mathbf{i}}{2} \operatorname{div}(\nabla \psi \bar{u} \partial_t u) + \mathbf{i} \partial_t (u \nabla \psi \nabla \bar{u}) \\ & \quad - \mathbf{i} \left( \frac{1}{2} \bar{u} \nabla \psi \nabla \partial_t u + \frac{1}{2} \partial_t u \nabla \psi \nabla \bar{u} + u \nabla \psi \nabla \partial_t \bar{u} \right). \end{aligned}$$

Taking the real part in the previous identity we get:

$$\begin{aligned} & \mathcal{R}e \mathbf{i} \partial_t u \left( \frac{1}{2} \Delta \psi \bar{u} + \nabla \psi \nabla \bar{u} \right) \\ &= \mathcal{R}e \mathbf{i} \left[ \frac{1}{2} \operatorname{div}(\nabla \psi \bar{u} \partial_t u) + \partial_t (u \nabla \psi \nabla \bar{u}) \right] \\ & \quad + \mathcal{I}m \left( \frac{1}{2} \bar{u} \nabla \psi \nabla \partial_t u + \frac{1}{2} \partial_t u \nabla \psi \nabla \bar{u} + u \nabla \psi \nabla \partial_t \bar{u} \right) \\ &= \mathcal{R}e \mathbf{i} \left[ \frac{1}{2} \operatorname{div}(\nabla \psi \bar{u} \partial_t u) + \partial_t (u \nabla \psi \nabla \bar{u}) \right] \\ & \quad + \frac{1}{2} \mathcal{I}m (\partial_t u \nabla \psi \nabla \bar{u} + u \nabla \psi \nabla \partial_t \bar{u}) \\ &= \mathcal{R}e \mathbf{i} \left[ \frac{1}{2} \operatorname{div}(\nabla \psi \bar{u} \partial_t u) + \partial_t (u \nabla \psi \nabla \bar{u}) \right] + \frac{1}{2} \mathcal{I}m \partial_t (u \nabla \psi \nabla \bar{u}) \end{aligned}$$

If we integrate this identity on the strip  $(-T, T) \times \mathbf{R}^n$  and we use the divergence theorem together with the assumptions done on the growth of the derivatives of  $\psi$ , then we get:

$$\begin{aligned} (1.3) \quad & \mathcal{R}e \int_{-T}^T \int_{\mathbf{R}^n} \mathbf{i} \partial_t u \left( \frac{1}{2} \Delta \psi \bar{u} + \nabla \psi \nabla \bar{u} \right) dx dt \\ &= -\frac{1}{2} \mathcal{I}m \sum_{\pm} \int_{\mathbf{R}^n} \nabla \psi \nabla \bar{u}(\pm T, \cdot) u(\pm T, \cdot) dx. \end{aligned}$$

On the other hand we have:

$$\begin{aligned}
& \mathcal{R}e \left[ \Delta u \left( \nabla \psi \nabla \bar{u} + \frac{1}{2} \Delta \psi \bar{u} \right) \right] \\
&= \mathcal{R}e \left[ \operatorname{div} (\nabla u (\nabla \psi \nabla \bar{u})) - \nabla u \nabla (\nabla \psi \nabla \bar{u}) \right. \\
&\quad \left. + \frac{1}{2} \operatorname{div} (\nabla u (\Delta \psi (x) \bar{u})) - \frac{1}{2} \nabla u \nabla (\bar{u} \Delta \psi) \right] \\
&= \mathcal{R}e \left[ \operatorname{div} (\nabla u (\nabla \psi \nabla \bar{u})) + \frac{1}{2} \operatorname{div} (\nabla u (\Delta \psi \bar{u})) \right. \\
&\quad \left. - \nabla u D^2 \bar{u} \nabla \psi - \nabla u D^2 \psi \nabla \bar{u} - \frac{1}{2} |\nabla u|^2 \Delta \psi - \frac{1}{2} \bar{u} (\nabla u \nabla (\Delta \psi)) \right] \\
&= \mathcal{R}e \left[ \operatorname{div} (\nabla u (\nabla \psi \nabla \bar{u})) + \frac{1}{2} \operatorname{div} (\nabla u (\Delta \psi \bar{u})) \right. \\
&\quad \left. - \frac{1}{2} \nabla (|\nabla u|^2) \nabla \psi - \nabla u D^2 \psi \nabla \bar{u} - \frac{1}{2} |\nabla u|^2 \Delta \psi - \frac{1}{4} \nabla (|u|^2) \nabla (\Delta \psi) \right] \\
&= \mathcal{R}e \left[ \operatorname{div} (\nabla u (\nabla \psi \nabla \bar{u})) + \frac{1}{2} \operatorname{div} (\nabla u (\Delta \psi (x) \bar{u})) \right. \\
&\quad \left. - \frac{1}{2} \operatorname{div} (|\nabla u|^2 \nabla \psi) + \frac{1}{2} |\nabla u|^2 \Delta \psi - \nabla u D^2 \psi \nabla \bar{u} \right. \\
&\quad \left. - \frac{1}{2} |\nabla u|^2 \Delta \psi - \frac{1}{4} \operatorname{div} ((|u|^2) \nabla (\Delta \psi)) + \frac{1}{4} |u|^2 \Delta^2 \psi \right].
\end{aligned}$$

If we integrate this identity on the strip  $(-T, T) \times \mathbf{R}^n$  and we use the divergence theorem as above, then we get

$$\begin{aligned}
(1.4) \quad & \mathcal{R}e \int_{-T}^T \int_{\mathbf{R}^n} \Delta u \left( \nabla \psi \nabla \bar{u} + \frac{1}{2} \Delta \psi \bar{u} \right) dx dt \\
&= \int_{-T}^T \int_{\mathbf{R}^n} \left( -\nabla u D^2 \psi \nabla \bar{u} + \frac{1}{4} |u|^2 \Delta^2 \psi \right) dx dt.
\end{aligned}$$

As a consequence of the identities (1.3) and (1.4), we can deduce the following one:

$$\begin{aligned}
(1.5) \quad & \int_{-T}^T \int_{\mathbf{R}^n} \left( \nabla \bar{u} D^2 \psi \nabla u - \frac{1}{4} |u|^2 \Delta^2 \psi \right) dt dx \\
&= -\frac{1}{2} \mathcal{I}m \sum_{\pm} \pm \int_{\mathbf{R}^n} u(\pm T, \cdot) \nabla \psi \nabla \bar{u}(\pm T, \cdot) dx.
\end{aligned}$$

By taking the limit as  $T \rightarrow \infty$  in (1.5) and by using lemma 1.1, we can deduce the desired result.  $\square$

## 2. APPLICATIONS

This section is devoted to the proof of corollary 0.1.

In order to do that we shall need the following lemma.

**Lemma 2.1.** *Assume that  $f \in \mathcal{S}(\mathbf{R}^n)$  is such that:*

$$(2.1) \quad \hat{f}(\xi) = 0 \quad \forall \xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}_\xi^n \text{ s.t. } |\xi_1| < \epsilon_0 = \epsilon_0(f) \text{ where } \epsilon_0 > 0.$$

*If  $u(t, x)$  is the corresponding solution to (0.1), then:*

$$(2.2) \quad \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \frac{|\nabla_\tau u(t, x)|^2}{|x|} |\partial_r \phi_R(|x|)| dx dt = 0,$$

$$(2.3) \quad \lim_{R \rightarrow \infty} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} |u(t, x)|^2 |\Delta^2 \phi_R(|x|)| dx dt = 0,$$

where  $\phi(|x|) \in C^\infty(\mathbf{R}^n)$  is any radially symmetric function such that:

$$\partial_r \phi(|x|) \leq C, |\Delta^2 \phi(|x|)| \leq \frac{C}{(1 + |x|)^3} \quad \forall x \in \mathbf{R}^n,$$

and  $\phi_R(|x|) := R\phi\left(\frac{|x|}{R}\right)$ .

**Proof.** Notice that (2.2) follows by combining the following implication:

$$u \text{ solves (0.1) with } f \in \mathcal{S}(\mathbf{R}^n) \Rightarrow \frac{|\nabla_\tau u|^2}{|x|} \in L^1(\mathbf{R}_t \times \mathbf{R}_x^n),$$

whose proof can be found in [5] (see also remark 0.4), with the following trivial fact:

$$\lim_{R \rightarrow \infty} \partial_r \phi_R(|x|) = \partial_r \phi(0) = 0 \quad \forall x \in \mathbf{R}^n,$$

where we have used the radially of  $\phi(|x|)$ .

Next we shall show (2.3). Let us introduce the unique function  $g \in \mathcal{S}(\mathbf{R}^n)$  such that:

$$\hat{g}(\xi) = \frac{\hat{f}(\xi)}{2\pi i \xi_1} \quad \forall \xi \in \mathbf{R}_\xi^n$$

and let us consider the unique solution  $v(t, x)$  to (0.1) with initial datum given by  $g(x)$ .

It is easy to check that  $\partial_{x_1} v(t, x) = u(t, x)$ . We can then apply (0.2) to the solution  $v(t, x)$  in order to get:

$$(2.4) \quad \begin{aligned} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |u(t, x)|^2 dx dt &= \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\partial_{x_1} v(t, x)|^2 dx dt \\ &\leq C \|g\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2 < \infty \quad \forall R > 0. \end{aligned}$$

Notice that due to the assumption done on  $\phi(|x|)$  we get:

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} |u(t, x)|^2 |\Delta^2 \phi_R(|x|)| dx dt &= \frac{1}{R^3} \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} |u(t, x)|^2 \left| \Delta^2 \phi\left(\frac{|x|}{R}\right) \right| dx dt \\ &\leq C \left( \frac{1}{R^3} \int_{-\infty}^{\infty} \int_{|x| < 1} |u(t, x)|^2 dx dt + \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{2^j < |x| < 2^{j+1}} \frac{|u(t, x)|^2}{(2^j + R)^3} dx dt \right) \end{aligned}$$

$$\leq C \left( \frac{1}{R^3} \int_{-\infty}^{\infty} \int_{|x|<1} |u(t, x)|^2 dx dt + \sum_{j=1}^{\infty} \int_{-\infty}^{\infty} \int_{|x|<2^{j+1}} \frac{|u(t, x)|^2}{(2^j + R)^3} dx dt \right),$$

that due to (2.4) implies:

$$(2.5) \quad \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} |u(t, x)|^2 |\Delta^2 \phi_R(|x|)| dx dt \leq C \left( \frac{1}{R^3} + \sum_{j=0}^{\infty} \frac{2^{j+1}}{(2^j + R)^3} \right).$$

On the other hand it is easy to show that

$$\lim_{R \rightarrow \infty} \frac{1}{R^3} + \sum_{j=0}^{\infty} \frac{2^{j+1}}{(2^j + R)^3} = 0,$$

that in conjunction with (2.5) implies (2.3).  $\square$

*Remark 2.1.* Let us notice that (2.3) follows, at least in dimension  $n \geq 4$ , from the following inequality:

$$(2.6) \quad \int_{-\infty}^{\infty} \int_{\mathbf{R}^n} \frac{|u|^2}{|x|^3} dx dt \leq C \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2 \quad \forall n \geq 4$$

whose proof can be found in [5].

**Proof of corollary 0.1.** First of all we show that (0.6) implies (0.7):

$$\begin{aligned} \sup_{R \in (0, \infty)} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\nabla u|^2 dx dt &\geq \sup_{R \in (0, \infty)} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\partial_r u|^2 dx dt \\ &\geq \lim_{R \rightarrow \infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\partial_r u|^2 dx dt = 2\pi \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2. \end{aligned}$$

On the other hand if we assume (0.8), then by (0.6) we get:

$$\begin{aligned} 0 &= \liminf_{R \rightarrow \infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\nabla u|^2 dx dt \geq \liminf_{R \rightarrow \infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\partial_r u|^2 dx dt \\ &= \lim_{R \rightarrow \infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\partial_r u|^2 dx dt = 2\pi \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2, \end{aligned}$$

then  $f = 0$  and in particular  $u(t, x) = 0$ .

Next we shall prove (0.6) assuming that the initial datum  $f$  is such that  $f \in \mathcal{S}(\mathbf{R}^n)$  and moreover it satisfies condition (2.1). It is easy to show, by combining a density argument with (0.2), that this regularity assumption done on  $f$  can be removed.

For any  $k \in \mathbf{N}$  we fix a function  $h_k(r) \in C_0^\infty(\mathbf{R}; [0, 1])$  such that:

$$h_k(r) = 1 \quad \forall r \in \mathbf{R} \text{ s.t. } |r| < 1, \quad h_k(r) = 0 \quad \forall r \in \mathbf{R} \text{ s.t. } |r| > \frac{k+1}{k},$$

$$h_k(r) = h_k(-r) \quad \forall r \in \mathbf{R}.$$

Let us introduce the functions  $\psi_k(r), H_k(r) \in C^\infty(\mathbf{R})$ :

$$\psi_k(r) = \int_0^r (r-s) h_k(s) ds \quad \text{and} \quad H_k(r) := \int_0^r h_k(s) ds.$$



Notice that

$$(2.7) \quad \psi_k''(r) = h_k(r), \psi_k'(r) = H_k(r) \forall r \in \mathbf{R} \text{ and } \lim_{r \rightarrow \infty} \partial_r \psi_k(r) = \int_0^\infty h_k(s) ds.$$

Moreover an elementary computation shows that:

$$(2.8) \quad \Delta^2 \psi_k(|x|) = \frac{C}{|x|^3} \forall x \in \mathbf{R}^n \text{ s.t. } |x| \geq 2 \text{ and } n \geq 2,$$

where  $\Delta^2$  is the bilaplacian operator, while in the one dimensional case, i.e. for  $n = 1$ , we have:

$$\partial_x^4 \psi_k(|x|) = 0 \forall x \in \mathbf{R} \text{ s.t. } |x| \geq 2.$$

Thus the functions  $\phi = \psi_k(|x|)$  satisfy the assumptions of lemma 2.1 in any dimension  $n \geq 1$ .

In the sequel we shall need the rescaled functions

$$\psi_{k,R}(|x|) := R \psi_k\left(\frac{|x|}{R}\right) \forall x \in \mathbf{R}^n, k \in \mathbf{N} \text{ and } R > 0$$

and we shall exploit the following elementary identity:

$$\nabla \bar{u} D^2 \psi \nabla u = \partial_r^2 \psi(|x|) |\partial_r u|^2 + \frac{\partial_r \psi(|x|)}{|x|} |\nabla_\tau u|^2,$$

where  $\psi(|x|)$  is any regular radial function and  $u$  is another regular function.

By combining this identity with (0.5), where we choose  $\psi(|x|) = \psi_{k,R}(|x|)$ , and recalling (2.7) we get:

$$(2.9) \quad \begin{aligned} & \int_{-\infty}^\infty \int_{\mathbf{R}^n} \left[ \partial_r^2 \psi_{k,R} |\partial_r u|^2 + \frac{\partial_r \psi_{k,R}}{|x|} |\nabla_\tau u|^2 - \frac{1}{4} |u|^2 \Delta^2 \psi_{k,R} \right] dx dt \\ &= 2\pi \left( \int_0^\infty h_k(s) ds \right) \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2 \forall k \in \mathbf{N}, R > 0. \end{aligned}$$

By using (2.2) and (2.3) where we make the choice  $\phi(|x|) = \psi_k(|x|)$  we get

$$(2.10) \quad \lim_{R \rightarrow \infty} \int_{-\infty}^\infty \int_{\mathbf{R}^n} \left[ \frac{|\nabla_\tau u|^2}{|x|} \partial_r \psi_{k,R} - \frac{1}{4} \Delta^2 \psi_{k,R} |u|^2 \right] dt dx = 0 \forall k \in \mathbf{N}.$$

We can combine now (2.9) with (2.10) in order to deduce

$$(2.11) \quad \lim_{R \rightarrow \infty} \int_{-\infty}^\infty \int_{\mathbf{R}^n} \partial_r^2 \psi_{k,R} |\partial_r u|^2 dx dt = 2\pi \left( \int_0^\infty h_k(s) ds \right) \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2 \forall k \in \mathbf{N}.$$

On the other hand, due to the properties of  $h_k$ , we get

$$\begin{aligned} & \frac{1}{R} \int_{-\infty}^\infty \int_{B_R} |\partial_r u|^2 dx dt \leq \int_{-\infty}^\infty \int_{\mathbf{R}^n} \partial_r^2 \psi_{k,R} |\partial_r u|^2 dt dx \\ &= \frac{1}{R} \int_{-\infty}^\infty \int_{\mathbf{R}^n} h_k\left(\frac{|x|}{R}\right) |\partial_r u|^2 dt dx \leq \frac{1}{R} \int_{-\infty}^\infty \int_{B_{\frac{k+1}{k}R}} |\partial_r u|^2 dx dt \end{aligned}$$

that due to (2.11) implies:

$$(2.12) \quad \limsup_{R \rightarrow \infty} \frac{1}{R} \int_{-\infty}^\infty \int_{B_R} |\partial_r u|^2 dx dt \leq 2\pi \left( \int_0^\infty h_k(s) ds \right) \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)}^2$$

$$\leq \frac{k+1}{k} \liminf_{R \rightarrow \infty} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\partial_r u|^2 dx dt \quad \forall k \in \mathbf{N}.$$

Since  $k \in \mathbf{N}$  is arbitrary and since the following identity is trivially satisfied:

$$\lim_{k \rightarrow \infty} \int_0^{\infty} h_k(s) ds = 1,$$

we can deduce easily (0.6) by using (2.12).

The proof is complete. □

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